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AUTHORS' REPLY

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In spite of the style adopted in the comments made concerning our paper [1], we are sincerely thankful to the authors of the comments for giving us the opportunity to clarify some points concerning this work, which may not have been made clear enough, or not have been very well understood. We would like to start by placing this work in its general context, in order to focus on its main purpose, and to discuss the many points raised. Then, specific comments will be made concerning the other observations made in the Letter to the Editor.

In a series of works developed in the last two decades, based on an experimental programme on geometrically non-linear vibration, and its effect on structural fatigue life, initiated at the ISVR by Bennouna and White for beams [2, 3], and continued in the case of homogeneous and composite plates by Benamar *et al.* [4, 5], measurements have been made of the effects of non-linearity on the deflection shapes of these structures, which have been shown to be amplitude dependent, with a higher increase in the induced non-linear stresses, which have been shown to affect significantly the structural fatigue life [6]. In order to examine theoretically this aspect of non-linear vibration, a semi-analytical approach, denoted in what follows as the SAA, has been developed in references [7–9]. The main features of this approach, which was applied for the first time to the shell case in the paper concerned in this reply, may be presented as follows:

1. In most of the studies on non-linear plate and shell vibration problems, the common approach has been, as outlined for example in reference [10], to separate the time and space variables, (which is an assumption which does not necessarily hold in the non-linear case, as mentioned in the early work of Chu [11]), to assume a mode shape, or two or more modes if more complex solutions are investigated, based upon physical understanding, intuition or the linear solution. The product of the unknown time function with the assumed mode shape, or a series of products of unknown time functions by the two or more spatial modes assumed, is then substituted into the non-linear partial differential equation of motion, which results, by using Galerkin's approximation procedure, in one or more non-linear differential equations in time, including non-linear terms (cubic or quadratic or both). In the SAA, the time function is assumed to be given, and in most of the studies it was assumed to be harmonic, *and the dependence of the spatial function on the amplitude of vibration is*

investigated. Of course, assuming a harmonic time function is known to be a simplification, which excludes many aspects of non-linear vibration such as the non-linear forced response harmonic distortion, which have been carefully investigated both theoretically and experimentally in our previous publications [3, 5], and also other non-linear effects, such as those mentioned in the comments. However, this simplification was made *deliberately* in order to examine another non-linear effect, i.e. the deformation induced by geometrical non-linearity on the deflection shapes, which has been detected experimentally, and which is of crucial importance in engineering design, because of its consequences on the non-linear stress patterns, and the structural fatigue life estimate.

2. From the theoretical point of view, the SAA is different from the most frequently used approach, partially described above, called in the remainder of this reply for simplicity the classical approach, and denoted CA, based on a search for an approximate solution of the non-linear partial differential equations of motion, with various associated techniques, such as stress function choice, compatibility condition verification, and the approximated numerical solution. The CA has been used by Chu and Herman [12] for the plate case and by Chu [11] in the shell case, and continued and completed by Evensen [13] in the shell case, with a more appropriate choice of trial function, and has been followed by many others. The SAA is a type of an extension of the classical Rayleigh–Ritz method, used in linear theory, to the non-linear problem, based on the use of the expression of the non-linear strain energy, the application of Hamilton's principle and the expansion of the unknown spatial function as a series of chosen basic functions. Of course, from the fundamental point of view, the non-linear partial differential equations of motion used in the CA, are the Euler–Lagrange equations associated with the variational principle of Hamilton used in the SAA, and hence, both formulations are theoretically equivalent. However, since none of the approaches permits an exact mathematical solution to be obtained, which is still unknown, the solution obtained in each case is approximate and involves many specific assumptions and uses particular numerical techniques. It should then not be forgotten that each approach is directed towards predicting specific effects, has a limited domain of validity, and has its internal logic, which cannot be transposed to another approach. Frequently, systematic analytical comparison between different approximate approaches is not possible and yield meaningless comparaters. An example concerning non-linear free flexural vibration of a circular cylindrical shell is that corresponding to the method developed by Alturi [14], and that presented by Evensen in reference [15]. Concerning these two methods, it was stated by Evensen in reference [16, p. 146], that "it was by no means clear that one approach was any worse (or better) than the other". This seems to us normal in the modelling of very complex problems, such as those of non-linear vibration, because each approach highlights one or a few facets of the problem and is valid under specific conditions. Another example related to our subject: in Chu's paper on shell vibration [11], the expression of the transversal displacement W and that of the stress function F were assumed. Also, Evensen [13] made the observation that this choice of W and F induced the non-satisfaction of the continuity condition on the axial displacement V. In our paper [1], the stress function was not assumed, but both W and V have been assumed as series of functions which automatically satisfy the continuity condition, which is curiously said not to be satisfied in the criticism, because the authors of the criticism thought of the problem in terms of the classical approach. Also, it is worth noticing here that use of linear mode shapes as basic functions in the displacement function expansions has been shown previously to be very appropriate in many cases of beams and plates [4, 5, 7–9]. However, this choice of basic functions has sometimes created a confusion between the modal coupling occurring in the forced response, and the modal participation, to a given non-linear free mode shape, considering in the SAA [1]. This seems to be the

source of the confusion which has led the authors of the criticism to write that our sentence; "Unfortunately the results available in the shell vibration literature are based on the single-mode approach" would have been true, if only this were 1961. What was meant in this sentence by "a single mode approach" was "a single function representation of the spatial vibration mode". The necessity, in the CA, to assume a particular deflection shape has been felt by many authors to be a problematic point. For example, in a paper on non-linear vibration of cylindrical shells by Chen and Babcock [17], this necessity was described as "a handicap". Also, in the conclusion of a paper by Leissa on "Non-linear analysis of plate and shell vibrations" [10], it is stated that: "further developments in the field of non-linear vibration could be made via use of additional, independent functions *to represent the modes* (most studies to date have used single function representations of the vibration modes)". In a more recent paper, by Ganapathi and Varadan on "Large amplitude vibrations of circular cylindrical shells" [18], the advantage of the method presented was said to be that: "unlike in any approximate analytical approach, no conditions/restrictions have been placed on the choice of the mode shapes".

3. The third feature of the SAA is that it was applied to beams and plates step by step, following the logic adopted in linear modal analysis. First, free vibrations have been considered, leading to determination of the non-linear mode shapes of the structures considered. Then, the forced case was examined for beams [19, 20] and is now being examined for plates and shells by our team. It is true that the concept of a non-linear mode shape is not absolutely clear and universally accepted like its linear equivalent. However, it is becoming quite familiar in the literature (see references [4, 5, 7, 9, 21], Nayfeh's paper [22] and his references [1, 2, 9, 11, 14–16, 18–20]), is very useful for the qualitative understanding of the non-linear behaviour, and is expected to play an important role in the development of the "non-linear modal analysis theory" mentioned in reference [21].

In the light of this, it should be clear that the purpose of our paper, as its title indicates was *only* to determine the first and second non-linear mode shapes of the structure considered, and not to study the forced non-linear response, which would have obliged us, for a complete but not always necessary description, to include axisymmetric modes into the series expansion and to examine the presence (or the non presence) of the companion mode in the response. This has been clearly explained in the comments made by Evensen [23], in which it is stated: "*although free vibrations* (*which is the subject of our paper*) *involve only one asymmetric mode* (*i.e.* $\cos(n\theta)$), it has been shown both experimentally and theoretically that *the forced vibrations* involve both a driven mode and its companion mode ($w \sim \sin(n\theta)$)".

In the context described above, the paper commented upon was an attempt to extend the SAA, which has been shown to be successful for beams and plates, to shell-type structures, in order to investigate its possibilities, with respect to non-linear mode shapes and non-linear stress analysis. To our knowledge, this aspect has not been examined neither in old literature (see references [2–6] of your comments), nor in the recent literature [18, 24]. In both, the investigations were mainly focussed on the non-linear frequency–amplitude dependence. This may be attributed, among other possible reasons, to the results of the early experimental work of Evensen on non-linear vibration of rings [25], in which it was found that the form of the measured mode shape was amplitude independent. In our opinion, based on the expressions for the non-linear terms obtained in our paper (equation 28(b)), which were found to be all proportional to the coefficient $\beta = h/R$, this result may be attributed to the very small value of 0·00127 for β for the very thin ring tested by Evensen [25], but should be more significant for higher values of the coefficient β and higher values of the amplitude of vibration.

Initially in the application of the SAA, an ideal case was considered, which was that of a cylindrical shell of infinite length. It is obvious that a cylindrical shell of infinite length if an idealization, which does not exist in reality, and which may be considered as a simplified case used to test a new approach, or to represent a given physical case, satisfying some appropriate conditions. In our paper, this question has not been discussed in detail. The authors of the criticism have stated that "the model is suitable for rings". At the end of this reply, we will specify exactly under which physical conditions our model, with only the asymmetric displacement functions used, may be applied to rings.

Concerning the type of non-linearity (softening or hardening) of cylindrical shells, this has been a subject of a large discussion in the literature [23, 26–29]. Both softening and hardening, and also softening followed by hardening behaviour, have been mentioned in the literature [18, 24, 30], depending on the shell geometrical characteristics, the mode wave numbers, the boundary conditions, and the amplitude of vibration. A complete review of this question would have exceeded the scope of the paper commented upon [1], which was not a review paper, and would exceed the scope of this reply. We would like to mention only that in a relatively recent paper by Ganapathi and Varadan [18], the study has been, to a great extent, "focussed on clarifying *existing controversies* about the non-linear behavior of thin circular cylindrical shells". It should be also noticed that the observation made in reference [17], according to which: "very few experimental studies have been devoted to non-linear vibration of shells" is still true to a great extent. Also, the few experimental works reported in the literature correspond to special geometrical characteristics, boundary conditions, amplitudes of vibration, and mode wave numbers. In our opinion, this is still insufficient for clarifying completely the fascinating, very rich, and often surprising subject of the non-linear behaviour of shells.

Returning now to the model proposed in our paper, it is true that it is suitable for rings. Moreover, the displacement functions used, *excluding the axisymmetric modes*, are suitable for the non-linear constrained vibration of rings. This case has been considered in a paper published by Evensen [31], entitled: "Non-linear vibration of an infinitely long cylindrical shell" and reported in the monograph of Leissa on shell vibration [32], in which the amplitude of the axisymmetric mode was taken to be zero, and "*a strongly non-linear behaviour of the hardening type*" was obtained (reference [31], p. 1402). This behaviour has been explained by the fact that, since the shell is constrained to vibrate with zero displacement at the nodes of cos (ny/R), "this vibration mode results in considerable stretching of the midsurface of the shell, which causes the strong non-linearity of the hardening type" (cf. [31], p. 1403). Notice that it was specified in the abstract of our paper that the considerable stretching was the origin of the non-linearity obtained. Experimentally we think that such a situation may be achieved practically by a technique similar to that mentioned in reference [31], in which the companion mode was constrained by "lightly placing a sharp point at the anti-node of the companion mode $\sin(n\theta)$ and then taking the measurements at the anti-node of the driven mode $\cos(n\theta)$". In the present case, the axisymmetric mode may be constrained by placing a sharp point at the nodes of the transversal displacements $\cos(n\theta)$, and measurements of the non-linear mode shape could be made at other points along the ring circumference. In a paper which was ready for submission to JSV when the comments arrived, further results will be given concerning the first four transverse non-linear mode shapes of shells of infinite length, and their corresponding non-linear stress patterns, with application to shells of finite length, considered at large circumferential wave number.

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